§6.4

16. Prove Cayley-Hamilton theorem for a complex $n\times n$ matrix A.
 If $f(t)$ is the characteristic polynomial of A, prove that $f(A)=0$

Proof: Schur's theorem $\Rightarrow$ $A=P^{-1}BP$ (B upper triangular P invertible)

$$f(t)=\prod_{i=1}^{n}(B_{ii}-t)$$

$C:=f(B)$ $\{e_i\}$ standard basis

$Ce_1=0$ $\left((B_{11}I-B)e_1=0 \right)$

$Ce_i=0$ $\Big((B_{ii}I-B)e_i$ is a linear combination of $e_1 \cdots e_{i-1}$ $\}$

$$\prod_{j=1}^{i-1}(B_{jj}I-B)\ (B_{ii}I-B)e_i=0 \Big)$$

$\Rightarrow$ $C=0$ i.e. $f(B)=0$

$f(A)=0$

18. $T: V\rightarrow W$ linear transformation between V and W, fin dim inner product space

Prove:
a) T^*T and TT^* are positive semi-definite.
b) $\operatorname{rank}(T^*T)=\operatorname{rank}(TT^*)=\operatorname{rank} T$

Proof: a) T^*T and TT^* are self-adjoint

Suppose $T^*Tx=\lambda x$.

$\lambda\langle x,x\rangle=\langle T^*Tx, x\rangle=\langle Tx, Tx\rangle\geq 0$

$\Rightarrow$ T^*T is positive semi-definite (cf. Ex 6.4.17 a))

Similarly for TT^*

b). To prove: $N(T^*T)=N(T)$.

- $x\in N(T^*T)$ $\Rightarrow$ $\langle T^*Tx, x\rangle=\langle Tx, Tx\rangle=0$ $\Rightarrow$ $Tx=0$

 i.e. $x\in NT$

- $x\in N(T)$ $\Rightarrow$ $T^*Tx=T^*0=0$ $\Rightarrow$ $N(T^*T)\supseteq N(T)$

Similarly $N(TT^*)=N(T)$.

- $\operatorname{rank} T = \operatorname{rank} T^*$

  $\operatorname{rank}([T]_\beta) = \operatorname{rank}([T]_\beta^*) = \operatorname{rank}([T^*]_\beta)$ for some orthonormal basis β

- $\Rightarrow \operatorname{rank} T^*T = \operatorname{rank} T = \operatorname{rank} T^* = \operatorname{rank} TT^*$

§6.4

16. Find an example of a unitary operator U on an inner product space and a U-invariant subspace W s.t. $W^\perp$ is not U-invariant

Solution: V the vector space of all sequences σ in F ($F = \mathbb{R}$ or $\mathbb{C}$). s.t.

$\sigma(n) \neq 0$, for only finitely many positive integers n.

$e_n(k) := \delta_{n,k}$ (Kronecker delta)

$\{e_n\}$ is a basis for V.

Define a unitary operator U by

$$\begin{cases} U(e_{2i+1}) = e_{2i-1} & i > 0 \\ U(e_1) = e_2 & \\ U(e_{2i}) = U(e_{2i+2}) & i > 0 \end{cases}$$

$\|U(x)\| = \|x\|$ and U is surjective

(The inner product is defined as ~~$\sigma(n)\gamma(n)$~~ $\cdot \langle \sigma, \gamma \rangle = \sum_n \sigma(n)\gamma(n)$

This has finite non-zero terms thus is well defined

The norm is induced from this inner product)

U is unitary

$W := \operatorname{Span}\{e_2 \; e_4 \; e_6 \cdots\}$

$W^\perp = \operatorname{Span}\{e_1 \; e_3 \; e_5 \cdots\}$

W is U-invariant

$e_2 \notin U(W)$ $W^\perp$ is not U-invariant since $U(e_1) = e_2 \notin W^\perp$

§6.6

7. T a normal operator on a finite-dim Complex inner product space V

Use the spectral decomposition $\lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$ of T to prove:

a) If g is a polynomial, then $g(T) = \sum_{i=1}^{k} g(\lambda_i) T_i$

<u>Proof</u>: By Thm 6.25 c) $T_i T_j = \delta_{ij} T_j$, $g(T) = \sum_{i=1}^{k} g(\lambda_i) T_i$

b) If $T^n = T_0$ for some n, then $T = T_0$

<u>Proof</u>: If $T_0 = T^n = \sum_{i=1}^{k} \lambda_i^n T_i$

v_i arbitrary eigenvector for eigenvalue λ_i.

$0 = T_0(v_i) = \lambda_i^n v_i \quad \Rightarrow a_i^n = 0 \Rightarrow a_i = 0$ for all i.

$T = \sum \lambda_i T_i = T_0$

c) U a linear operator on V. Then U commutes with T iff U commutes with each T_i

<u>Proof</u>: • By Corollary 4 to Thm 6.25. $T_i = g_i(T)$ for some g_i: polynomial.

$\Rightarrow U$ commutes with T_i's ~~iff~~ if U commutes with T.

• Conversely, if U commutes with T_i's, then

$TU = (\sum_{i=1}^{k} \lambda_i T_i) U = \sum_{i=1}^{k} \lambda_i T_i U = \sum_{i=1}^{k} \lambda_i U T_i = U \sum_{i=1}^{k} \lambda_i T_i = UT$

d) $\exists$ a normal operator U on V s.t. $U^2 = T$

<u>Proof</u>: $U = \sum_{i=1}^{k} \lambda^{\frac{1}{2}} T_i$.

e) T is invertible iff $\lambda_i \neq 0$ for $1 \le i \le k$

<u>Proof</u>: T is invertible iff $N(T) = \{0\}$ iff 0 is not an eigenvalue

f) T is a projection iff every eigenvalue of T is 1 or 0

Solution: • If every eigenvalue of T is 1 or 0, then $T = 0T_0 + 1T_1 = T_1$.

• If T is a proj on W along W', then

$\forall\, v' \in V$ can be written as $u + v$, $u \in W$, $v \in W'$

If λ is an eigenvalue,

$$u = T(u+v) = \lambda(u+v)$$

$$(1-\lambda)u = \lambda v$$

$$\Rightarrow \lambda = 1 \text{ or } 0.$$

g) $T = -T^*$ iff every λ_i is an imaginary number.

Proof: $T^* = \sum_{i=1}^{k} \overline{\lambda_i T_i}$